A Hybrid Yee Algorithm/Scalar-Wave Equation Approach

Paul H. Aoyagi, Jin-Fa Lee, and Raj Mittra

***Abstract*—In this paper, two alternate formulations of the Yee algorithm, namely, the finite-difference time-domain (FDTD) vector-wave algorithm and the FDTD scalar-wave algorithm are examined and compared to determine their relative merits and computational efficiency. By using the central-difference divergence relation the conventional Yee algorithm is rewritten as a hybrid Yee/FDTD scalar-wave algorithm. It is found that this can reduce the computation time for many 3-D open geometries, in particular planar structures, by approximately two times as well as reduce the computer-memory requirements by approximately one-third. Moreover, it is demonstrated both mathematically and verified by numerical simulation of a coplanar strip transmission line that this hybrid algorithm is entirely equivalent to the Yee algorithm. In addition, an alternate but mathematically equivalent reformulation of the Enquist-Majda absorbing boundary condition based on the normal field component (relative to the absorbing boundary wall) is given to increase the efficiency of the hybrid algorithm in the modeling of open region problems. Numerical results generated by the hybrid Yee/scalar-wave algorithm for the Vivaldi antenna are given and compared with published experimental work.**

I. Introduction

In the past few years the Yee algorithm [1] has been demonstrated to be a viable technique for solving a variety of problems in electromagnetics [2]–[4]. Though there are other FDTD formulations that can potentially be used to solve Maxwell's equations, i.e., those based on the vector- and scalar-wave equations, there has been very little published work investigating these formulations and their relative merits. The purpose of this work is to study the FDTD scalar-wave and vector-wave algorithms and to compare each with the more conventional Yee algorithm to determine their respective computational efficiency. The major objectives of this work are: (i) to demonstrate the mathematical and numerical equivalence between the Yee algorithm and a FDTD vector-wave equation, (ii) to demonstrate the mathematical and numerical equivalence of the FDTD scalar-wave algorithm with the Yee algorithm for the time-domain modeling of 3-D divergence-free electromagnetic fields, (iii) to demonstrate that a hybrid Yee/scalar-wave FDTD algorithm can be combined with absorbing boundary conditions (ABC) to generate results numerically identical to the conventional Yee algorithm for coplanar microwave integrated circuits but at half the computation and one-third the computer memory, and (iv) to compare the numerical results generated by the hybrid Yee/scalar-wave algorithm for the Vivaldi antenna with published experimental work.

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P. Aoyagi and R. Mittra are with the Department of Electrical and Computer Engineering, University of Illinois at Urbana-Champaign, Urbana, IL 61801.
J.-F. Lee is with the Department of Electrical and Computer Engineering, Worcester Polytechnic Institute, Worcester, MA 01609.

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II. Different Types of FDTD Formulations

A. The Yee Algorithm

The Yee algorithm is a central difference approximation of Maxwell's curl equations written in explicit form. The relative locations of the electric- and magnetic-field components in a uniform, Cartesian grid is defined by the so-called Yee lattice (Fig. 1). Typical examples of an electric- and magnetic-field finite-difference equations for lossless media (excluding the source term) are given by

$$\begin{aligned} E_x^{l+1}\left(m+\frac{1}{2},n,p\right) &= E_x^{l}\left(m+\frac{1}{2},n,p\right) + \frac{\Delta t}{\varepsilon(m+\frac{1}{2},n,p)\Delta s} \\ &\times\left[H_z^{l+1/2}\left(m+\frac{1}{2},n+\frac{1}{2},p\right) - H_z^{l+1/2}\left(m+\frac{1}{2},n-\frac{1}{2},p\right)\right. \\ &+ H_y^{l+1/2}\left(m+\frac{1}{2},n,p-\frac{1}{2}\right) \\ &\left. - H_y^{l+1/2}\left(m+\frac{1}{2},n,p+\frac{1}{2}\right)\right] \end{aligned} \tag{1}$$

$$\begin{aligned} H_y^{l+1/2}\left(m+\frac{1}{2},n,p+\frac{1}{2}\right) &= H_y^{l-1/2}\left(m+\frac{1}{2},n,p+\frac{1}{2}\right) + \frac{\Delta t}{\mu(m+\frac{1}{2},n,p+\frac{1}{2})\Delta s} \\ &\times\left[E_z^{l}\left(m+1,n,p+\frac{1}{2}\right) - E_z^{l}\left(m,n,p+\frac{1}{2}\right) + E_x^{l}\left(m+\frac{1}{2},n,p\right)\right. \\ &\left. - E_x^{l}\left(m+\frac{1}{2},n,p+1\right)\right] \end{aligned} \tag{2}$$

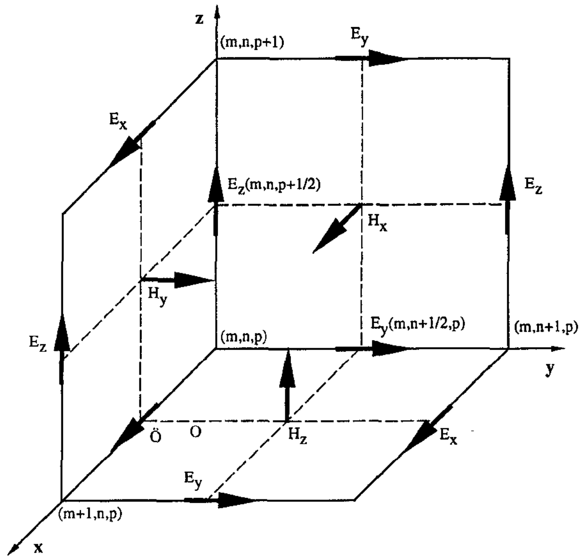

Fig. 1. The Yee lattice depicting the relative spacial positions of the electric and magnetic fields corresponding to a central-differencing of the Yee algorithm as well as the FDTD vector-wave and scalar-wave algorithms.

where (m, n, p) are position indices defined such that $x = m\Delta x, y = n\Delta y, z = p\Delta z$

$\Delta s = \Delta x = \Delta y = \Delta z$ (uniform space discretization)

l = time index such that $t = l\Delta t$

Δt = time discretization

ε = electric permitivitty

μ = magnetic permeability.

An important characteristic of these equations is that they are coupled, i.e., one cannot compute any single field component without having to compute other field components. Consequently, six field components per cell must be stored and a minimum of 24 additions and 6 multiplications per cell must be performed in order to advance one time step, Δt, where a "cell" is defined to be a unit volume $\Delta s x \Delta s x \Delta s$.

B. The Vector-Wave Equation

By considering the six algebraic equations of the Yee algorithm it is apparent that there is a redundancy built into its formulation. In particular, by considering all six difference equations it is possible to substitute all of the magnetic-field expressions in (1) solely in terms of electric fields and obtain (3), which is shown at the bottom of the page where $v^2 = c^2/[\varepsilon_r(m + 1/2, n, p)u_r(m + 1/2, n, p)]$, and ε_r, u_r are relative permitivity and permeability, respectively. By performing similar manipulations on the E_y and E_z equations the necessary number of difference equations per cell can be reduced from 6 to 3. It can be readily shown that the resulting equations are exactly equivalent to the central difference approximation of the vector-wave equation written in explicit form, i.e.

$$\nabla x \nabla x \bar{E} + \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\bar{E} = 0. \tag{4}$$

Because the FDTD vector-wave equation can be derived directly from the Yee algorithm through algebraic manipulation (analogous to performing continuous vector operations on Maxwell's curl equations) the results generated by both difference schemes will be numerically identical. However, despite having eliminated the explicit computation of the magnetic fields, (3) shows that the FDTD vector-wave formulation will actually increase rather than decrease the minimum number of additions required per iteration from 24 to 39 per cell. Moreover, because values at the time index l and $l - 1$ must be stored, the memory requirements are actually no better than those of the Yee algorithm and at the cost of losing field information. It can be concluded that a fully explicit FDTD vector-wave formulation can provide little if any practical advantage over the conventional Yee algorithm.

C. The Scalar-Wave Equation

The Yee algorithm and the FDTD vector-wave algorithm, however, can be simplified further, provided the fields are assumed or known to be locally divergence-free. Mathematically,

$$\begin{aligned} E_x^{l+1}\left(m+\frac{1}{2}, n, p\right) = {} & \left[2 - 4\left(\frac{v(m+\frac{1}{2}, n, p)\Delta t}{\Delta s}\right)^2\right] \\ & \cdot E_x^{1}\left(m+\frac{1}{2}, n, p\right) - E_x^{l-1}\left(m+\frac{1}{2}, n, p\right) \\ & + \left(\frac{v(m+\frac{1}{2}, n, p)\Delta t}{\Delta s}\right)^2 \\ & \cdot \left\{ \begin{array}{l} E_x^l(m+\frac{1}{2}, n+1, p) + E_x^l(m+\frac{1}{2}, n-1, p) + E_x^l(m+\frac{1}{2}, n, p+1) \\ + E_x^l(m+\frac{1}{2}, n, p-1) - E_y^l(m+1, n+\frac{1}{2}, p) + E_y^l(m+1, n-\frac{1}{2}, p+1) \\ - E_z^l(m+1, n, p+\frac{1}{2}) + E_z^l(m+\frac{1}{2}, n, p-\frac{1}{2}) + E_y^l(m, n+\frac{1}{2}, p) \\ - E_y^l(m, n-\frac{1}{2}, p) + E_z^l(m, n, p+\frac{1}{2}) - E_z^l(m, n, p-\frac{1}{2}) \end{array} \right\} \end{aligned} \tag{3}$$

this is equivalent to assuming that the fields satisfy the central-difference approximation of the divergence relation. Assuming that $\Delta x = \Delta y = \Delta z$, this relation can be expressed in terms of the Yee lattice as

$$E_x^l\left(m+\frac{1}{2},n,p\right) - E_x^l\left(m-\frac{1}{2},n,p\right) + E_y^l\left(m,n+\frac{1}{2},p\right) - E_y^l\left(m,n-\frac{1}{2},p\right) + E_z^l\left(m,n,p+\frac{1}{2}\right) - E_z^l\left(m,n,p-\frac{1}{2}\right) = 0. \quad (4)$$

By substituting (5) into (3) expression (6), which is shown at the bottom of the page, can be obtained.

By considering (6) as well as the analogous expressions for the E_y and E_z fields, it can be shown that the resulting difference equations are identical to the central-difference approximation of the wave equation written in explicit form, i.e.

$$\nabla^2 \bar{E} + \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\bar{E} = 0. \quad (6)$$

In contrast to the FDTD vector-wave equation given by (3), (6) shows that a FDTD scalar-wave-equation formulation can result in a modest computational savings over the Yee algorithm. In particular, it can be seen that the FDTD scalar-wave algorithm will require 21 additions and six multiplications per cell, per iteration to generate the total electric-field transient response as opposed to 24 addition and six multiplications per cell, per iteration required by the Yee algorithm. If free space is being modeled, the computational requirements of the FDTD scalar-wave algorithm actually decrease slightly more to just 18 additions and three multiplications per cell, per iteration provided the time step is set equal to the Courant, Friedrichs, Lewy (CFL) stability limit. The greatest economy, however, of using a FDTD scalar-wave formulation will result from the flexibility of the formulation. In particular, unlike the Yee algorithm and the FDTD vector-wave formulation, the FDTD scalar-wave formulation consists of six uncoupled algebraic equations which, in turn, implies that any single field component can be computed without necessarily having to pay the overhead of computing other field components. Consequently, the number of computations can be decreased further by omitting all but the most essential or desired field components from the formulation. Moreover, because it can be shown that an initial-value problem based on Maxwell's source-free curl equations and the scalar-wave equations will preserve the divergence-free nature of the fields at time is advanced (see Appendixes I and II), the FDTD scalar-wave algorithm can be expected to generate numerical results identical to those of the Yee algorithm, provided the fields and their initial condition are known to be divergence-free.

III. Hybrid FDTD Formulations

A. Combining the Scalar-Wave Equation with the Vector-Wave Equation

Though there can be significant computational advantage, it is important to emphasize the fact that the FDTD scalar-wave algorithm will be valid if and only if the fields are known to be divergence-free for all time, and this imposes a limit on its usage and generality that would normally not apply to the Yee algorithm of the FDTD vector-wave algorithm. Consequently, one must be careful to determine whether the scalar-wave equation is applicable in a given situation. One simply way, however, to extend the applicability of the FDTD scalar-wave formulation is to combine it with the FDTD vector-wave algorithm. By applying the FDTD vector-wave algorithm locally to the nondivergence free regions of the problem domain such as pec edges, dielectric interfaces, and/or sources, and applying the more economical FDTD scalar-wave algorithm to the remaining divergence-free regions,. i.e., those regions that have no discontinuities, the two algorithms can be combined to generate the time-domain response of any isotropic, inhomogeneous scattering problem.

To demonstrate the feasibility of this hybrid formulation, we consider modeling the time-domain response of a time-harmonic electric-field source radiating inside a $75\Delta s \times 75\Delta s \times 75\Delta s$ isotropic, homogeneous pec/air cavity using a uniform, cubic mesh. For computational efficiency the time step was set equal to the well-known CFL stability limit. The source was chosen to be a single E_x field with a time-harmonic dependence located near the center of the cavity. The problem domain was partitioned into two regions (Fig. 1). In region 1 a FDTD (electric field) vector-wave algorithm was applied to model the nondivergence-free fields around the source. Region 2 consisted of the remaining divergence-free volume where a FDTD (electric field) scalar-wave algorithm was used. The resulting electric-field distributions were then compared with those of the conventional Yee algorithm. It was found that the hybrid algorithm ran approximately 1.37 times faster than the conventional Yee algorithm (730 compared to 1000 CPU seconds) for comparably vectorized computer programs on the Cray-YMP. Moreover, it was found that both algorithms generated numerical results identical to within

$$E_x^{l+1}\left(m+\frac{1}{2},n,p\right) = \left(2-6\left(\frac{v\Delta t}{\Delta s}\right)^2\right)E_x^l\left(m+\frac{1}{2},n,p\right) - E_x^{l-1}\left(m+\frac{1}{2},n,p\right) + \left(\frac{v\Delta t}{\Delta s}\right)^2 \left\{ \begin{array}{l} E_x^l(m+\frac{1}{2},n+1,p) + E_x^l(m+\frac{1}{2},n-1,p) + E_x^l(m+\frac{1}{2},n,p+1) \\ +E_x^l(m+\frac{1}{2},n,p-1) - E_x^l(m+\frac{3}{2},n,p) + E_x^l(m-\frac{1}{2},n,p) \end{array} \right\}. \quad (5)$$

nine decimal places over the entire problem domain even after 10 000 iterations, demonstrating not only equivalence but stability of the algorithm.

B. Combining the Scalar-Wave Equation with the Yee Algorithm

Having verified the numerical equivalence between the FDTD scalar-wave/vector-wave hybrid formulations with the conventional Yee algorithm, we note that the computational efficiency could have been increased even further by partitioning the problem domain so that the divergence-free regions can be interfaced with the nondivergence free regions without having to use all three electric- (or all magnetic-) field components to interface region 1 with region 2. Though more elaborate partitioning schemes are possible, a simple and easily vectorizable scheme that can be used to reduce the number of field components is to partition the problem into planar volumes, which makes the present hybrid approach very attractive in modeling planar and/or coplanar microwave integrated circuits. Since the interface between each region is 2-D in nature, only the tangential electric (or tangential magnetic) field on that interface are needed to model the fields in each divergence-free region. Consequently, only two out of the three electric (or magnetic) FDTD scalar-wave equations in a Cartesian system are needed to model the divergence-free regions, reducing the memory requirements of the FDTD scalar-wave algorithm from six fields per cell to four fields per cell and, more significantly, the number of computations per iteration from 24 additions, six multiplications to just 14 additions, four multiplications per cell assuming a uniform cubic mesh is used. For the special case of modeling free space, the number of additions needed to implement the FDTD scalar-wave formulation can be reduced even further to just 12 additions, two multiplications by choosing the time step to be near the stability limit. Table I summarized and compares all of the computational and memory requirements of the Yee algorithm, the FDTD vector-wave and scalar-wave algorithms.

To demonstrate the economy of using the hybrid Yee/scalar-wave formulation with planar partitioning as well as its equivalence to the standard Yee algorithm, a coplanar strip (CPS) transmission line inside a $75\Delta s \times 75\Delta s \times 75\Delta s$ pec cavity partitioned into three planar volumes was simulated using a time-harmonic electric-field source (Fig. 2). Region 2 was chosen to be a planar volume approximately two cells thick encompass the CPS line on thin dielectric substrate ($\varepsilon_r = 5.0$). Regions 1 and 3 were chosen to consist of the remaining divergence-free volumes. A FDTD scalar-wave algorithm was applied to E_y, E_z in regions 1 and 3 while the conventional Yee algorithm was applied in region 2. Both simulations were run at the maximum time step allowed for by the CFL stability condition using a uniform, cubic mesh. After 10 000 iterations, it was found that the E_y and E_z field distributions over the entire volume were identical to that obtained using a full Yee algorithm up to 10 decimal places on the Cray YMP. Moreover, the computation time of the hybrid formulation was found to be approximately 1.54 times faster than the Yee algorithm for comparably vectorized codes, i.e., 648 seconds compared to 1000 seconds. We note that because the Yee algorithm can be vectorized extremely efficiently when pec walls are used, as opposed to an absorbing boundary wall, the improvement in computer time falls somewhat short of the theoretical limit of 2.14 implied by Table I.

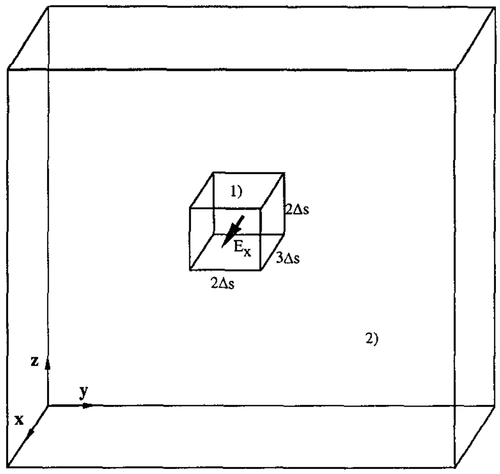

Fig. 2. Partitioning of a hybrid FDTD algorithm used to model the time-domain response of a $25\Delta s \times 25\Delta s \times 25\Delta s$ pec cavity using a uniform, cubic grid. The FDTD (electric field) vector-wave was applied around a time-harmonic electric-field source located near the center of the cavity, i.e., region 1, while the FDTD (electric field) scalar-wave algorithm was applied to region 2.

TABLE I
COMPARISON OF THE MINIMUM COMPUTATION PER ITERATION AND MEMORY COSTS OF THE VARIOUS 3-D FDTD TECHNIQUES THE * INDICATES THE COSTS ASSOCIATED WITH THE FREE-SPACE SIMULATIONS OPERATED AT THE CFL STABILITY LIMIT

method	add/cell	multi./cell	memory/cell
Yee algorithm	24	6	6
FDTD vector-wave	36	6	6
FDTD scalar-wave (3 field fomulation)	21(18)*	6(3)*	6
FDTD scalar-wave (2 field formulation)	14(12)*	4(2)*	4

C. Application of an Absorbing Boundary Condition

Unlike the previous examples, many transient electromagnetics problems require the application of an absorbing boundary condition (ABC). Typically, when using the Yee algorithm in a Cartesian system, the ABC is applied directly to the two field components tangential to the ABC wall. Indeed, strictly speaking, many of the ABC's are valid only when applied in this manner. If a FDTD scalar-wave formulation is used, however, this straightforward application of the ABC is not always possible. If, as in the previous example, a FDTD scalar-wave algorithm is applied using only the E_y and E_z components, there will be difficulties in applying an ABC to an xy- and xz-plane because only one tangential field component rather than two will be available for computation. This leads to a very novel predicament. In particular, though the ABC can be applied to the known tangential component independently of the unknown tangential component (in a Cartesian system), there is sufficient information to compute the normal field components interior to the ABC wall without the other tangential component. One way to remedy this is to apply an ABC in some fashion to the normal field component. Though there are several possible ways to accomplish this, an approach mathematically identical to the conventional finite-difference application of the ABC is to apply the ABC to the normal derivative of the normal field. To illustrate this equivalence, consider the conventional application of the first-order Engquist-Majda ABC to the tangential electric field components relative to a yz plane located at $x = m\Delta x$ [5], i.e.

$$E_y^{l+1}\left(m, n+\frac{1}{2}, p\right) = E_y^l\left(m-1, n+\frac{1}{2}, p\right) + \frac{v\Delta t - \Delta s}{v\Delta t + \Delta s} \cdot \left[E_y^{l+1}\left(m-1, n+\frac{1}{2}, p\right) - E_y^l\left(m, n+\frac{1}{2}, p\right)\right] \quad (7)$$

$$E_z^{l+1}\left(m, n, p+\frac{1}{2}\right) = E_z^l\left(m-1, n, p+\frac{1}{2}\right) + \frac{v\Delta t - \Delta s}{v\Delta t + \Delta s} \cdot \left[E_z^{l+1}\left(m-1, n, p+\frac{1}{2}\right) - E_z^l\left(m, n, p+\frac{1}{2}\right)\right]. \quad (8)$$

By applying the ABC separately to adjacent field components and superimposing the equations, it can be shown that

$$G_{yz}^{l+1}(m,n,p) = G_{yz}^l(m-1,n,p) + \frac{v\Delta t - \Delta s}{v\Delta t + \Delta s}\left[G_{yz}^{l+1}(m-1,n,p) - G_{yz}^l(m,n,p)\right] \quad (9)$$

where

$$G_{yz}^l(m,n,p) = -E_y^l\left(m, n+\frac{1}{2}, p\right) + E_y^l\left(m, n-\frac{1}{2}, p\right) - E_z^l\left(m, n, p+\frac{1}{2}\right) + E_z^l\left(m, n, p-\frac{1}{2}\right). \quad (10)$$

If the fields at the absorbing boundary are divergence-free, i.e., the fields satisfy (5), G_{yz}^l will be mathematically equivalent to the central-difference approximation of the normal derivative of the normal field, i.e.

$$G_{yz}^l(m,n,p) = G_x^l(m,n,p) \quad (11)$$

where

$$G_x^l(m,n,p) = E_x^l\left(m+\frac{1}{2}, n, p\right) - E_x^l\left(m-\frac{1}{2}, n, p\right). \quad (12)$$

By computing G_x^l on the ABC wall the interior normal fields can be computed at a future time step using the FDTD vector-wave equation. Because of the mathematical equivalence between the Yee algorithm and the FDTD vector-wave equation, application of the first-order Engquist-Majda ABC to the normal field information (in the form of G_x^l) with the FDTD vector-wave algorithm will be numerically equivalent to the ABC applied to the tangential field components with the Yee algorithm.

To verify the equivalence between the conventional application of the ABC with the proposed reformulation, the time-domain simulation of the coplanar strip transmission line (see Fig. 3) was repeated using the first-order Engquist-Majda ABC with the hybrid Yee/scalar-wave algorithm using a time-harmonic electric-field source. The G_x^l ABC formulation was applied with the FDTD scalar-wave algorithm in the divergence-free regions whereas the conventional tangential field ABC formulation was applied with the Yee algorithm in the nondivergence-free regions. After 10 000 iterations it was found that the E_y and E_z values generated by the hybrid algorithm over the entire volume are identical up to nine decimal places to those obtained using the full Yee algorithm with the conventional application of the ABC. Moreover, the computational savings of the hybrid algorithm was still found to be almost 2.1 times faster than the Yee algorithm (756 compared to 1593 CPU seconds) for comparably vectorized code.

Though we have considered the first-order Engquist-Majda ABC, similar analysis can also be used to show that the second-order Engquist-Majda ABC as well as the Liao ABC (or any order) [6] can also be equivalently reformulated.

IV. Computation of the E-Plane Radiation Pattern of the Vivaldi Antenna Using the Yee/Scalar-Wave Algorithm

As a practical demonstration of the Yee/scalar-wave algorithm, the far field E-plane free space radiation patterns of a Vivaldi antenna with and without dielectric substrate (Fig. 4) is computed and compared with published experimental results. Numerical simulations were run using a transient (gaussian) electric-dipole source located at the back of the antenna. The far-field was then computed using the equivalence principle in conjunction with the near- to far-field time-domain translation algorithm outlined in [7] and [8]. These time-domain results were then Fourier-transformed to obtain the frequency dependence of the radiation patterns. For computational efficiency, the far-field was computed using the smallest equivalence surface required to enclose the antenna. Interestingly, we note that in addition to computational efficiency, choosing the smallest equivalence surface will also increase the accuracy of the far-field computation by reducing the numerical-phase error introduced into the equivalent sources by the application of the finite-difference approximation.

We begin by comparing the numerical results generated by the hybrid Yee/scalar-wave algorithm with the experimental results for a Vivaldi without a dielectric substrate at a single operating frequency, f_0. The finite-difference discretization was chosen such that $\lambda_0 = 20\Delta s$, where λ_0 = free-space wavelength corresponding to f_0 and $\Delta s = \Delta x = \Delta y = \Delta z =$ cell size. The geometry of the Vivaldi was modeled using a stair-casing approximation and longitudinal (pec) symmetry abut the geometry was used to reduce the problem domain by one-half. Information regarding the specific dimensions of the

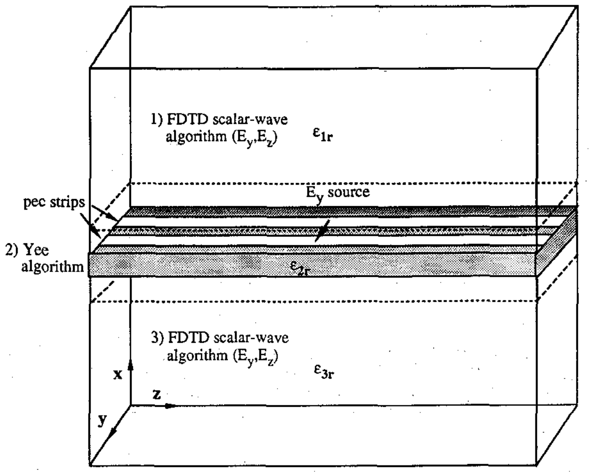

Fig. 3. Partitioning of a hybrid FDTD algorithm used to model the time-domain response of a coplanar strip (CPS) transmission line on a thin dielectric substrate ($\varepsilon_{1r} = 1.00, \varepsilon_{2r} = 5.00, \varepsilon_{3r} = 1.00$) inside a $25\Delta s \times 25\Delta s \times 25\Delta s$ uniform, cubic grid. The Yee algorithm was applied to a planar volume around the CPS line, i.e., region 2, while the FDTD (E_y, E_z) scalar-wave algorithm was applied to regions 1 and 3.

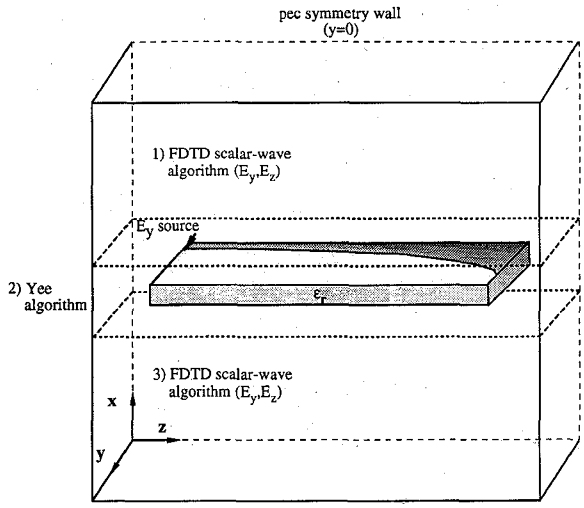

Fig. 5. Partitioning of the hybrid Yee/scalar-wave algorithm used to model the time-domain response of a Vivaldi antenna in free space. A pec xz-wall $(y = 0)$ is along the longitudinal axis of the Vivaldi splitting the structure in half. The Yee algorithm was applied to a thin planar volume around the antenna, i.e., region 2, while the FDTD (E_y, E_z) scalar-wave algorithm was applied to regions 1 and 3.

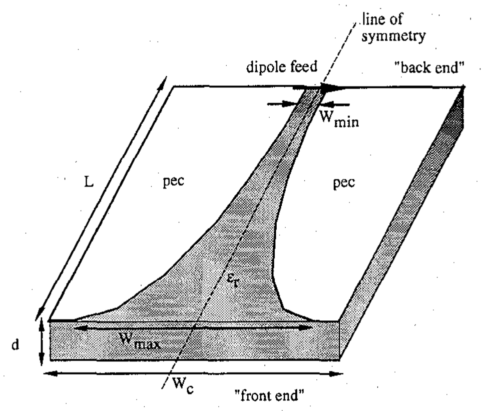

Fig. 4. The Vivaldi Antenna.

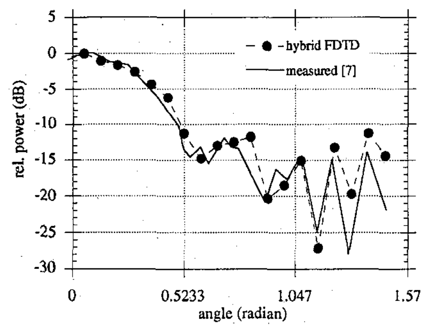

Fig. 6. Comparison of the hybrid Yee/scalar-wave algorithm results with published experimental data [9] for a Vivaldi antenna with no dielectric substrate.

Vivaldi are given in Table II. In an attempt to economically reduce unwanted interactions with the antenna, the ABC walls were placed approximately $0.5\lambda_0, 1.0\lambda_0, 1.25\lambda_0$, and $1.5\lambda_0$ from the back, front, top/bottom, and side of the Vivaldi, respectively. This resulted in a $50\Delta s \times 147\Delta s \times 157\Delta s$ problem domain divided into three planar regions: a small nondivergence-free region surrounding the Vivaldi ($20\Delta s \times 147\Delta s \times 157\Delta s$) and two remaining divergence-free regions (both $24\Delta s \times 147\Delta s \times 157\Delta s$). The conventional Yee algorithm was used to compute the transient response of the nondivergence-free region while a (E_y, E_z) FDTD scalar-wave algorithm was used to compute the transient response of the divergence-free region (Fig. 5). Total computation time of the hybrid algorithm was approximately 4 CPU minutes for 1200 iterations on the Cray YMP. Fig. 6 compares the results generated by the Yee/scalar-wave algorithm with the experimental results published in [9]. The results are found to be in good agreement.

We next consider modeling a Vivaldi with dielectric substrate ($\varepsilon_r = 2.22$) using the same Yee/scalar wave formulation. The new dimensions of the Vivaldi being modeled are given

TABLE II
PHYSICAL DIMENSIONS OF THE VIVALDI WITHOUT DIELECTRIC SUBSTRATE ($\varepsilon_r = 1.00$) USED TO COMPARE HYBRID YEE/SCALAR-WAVE ALGORITHM ($\Delta s = \Delta x = \Delta y = \Delta z$) SIMULATIONS WITH EXPERIMENT [9] ($f_0 = 10.0$ GHz, $\lambda_0 = 30$ mm $= 20\Delta s$)

geometric characteristic	dimensions [9]	finite-difference approximation
L	$6.30\lambda_0$	$126\Delta s$
$W_{max}/2$	$1.77\lambda_0$	$19\Delta s$
$W_c/2$	$5.10\lambda_0$	$103\Delta s$
$W_{min}/2$	$0.01\lambda_0$	$1\Delta s$

in Table III. In contrast to the previous case, it was found that a finer discretization, i.e., $\lambda_0 = 34\Delta s$, was needed to adequately model the dielectric thickness, the associated decrease in wavelength as well as the more dramatic flaring of the antenna. To accommodate this discretization and to allow for an ABC wall placement reasonable far away from the antenna, a problem domain $50\Delta s \times 86\Delta s \times 263\Delta s$ was

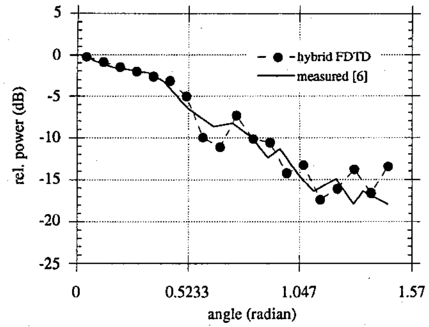

Fig. 7. Comparison of the hybrid Yee/scalar-wave algorithm results with published experimental data [10] for a Vivaldi antenna with dielectric substrate ($\varepsilon_r = 2.22$).

TABLE III
PHYSICAL DIMENSIONS OF THE VIVALDI WITH DIELECTRIC SUBSTRATE ($\varepsilon_r = 2.22$) USED TO COMPARE HYBRID YEE/SCALAR-WAVE ALGORITHM ($\Delta s = \Delta x = \Delta y = \Delta z$) SIMULATIONS WITH EXPERIMENT [10] ($f_0 = 35.0$ GHz, $\lambda_0 = 8.5\text{mm} = 34\Delta s$)

geometric characteristic	dimensions [10]	finite-difference approximation
d	$0.058\lambda_0$	$2\Delta s$
L	$0.677\lambda_0$	$232\Delta s$
$W_{max}/2$	$0.315\lambda_0$	$53\Delta s$
$W_c/2$	$0.315\lambda_0$ (approx.)	$53\Delta s$
$W_{min}/2$	$0.010\lambda_0$ (approx.)	$1\Delta s$

used in the numerical simulations. (We note, however, due to memory limitations, the ABC walls are approximately 20% closer relative to wavelength than the previous case.) As before, longitudinal symmetry was used to reduce memory and computation by one-half. Fig. 7 compares the E-plane pattern generated by the Yee/scalar wave algorithm with the measured results published in [10]. The total computation time of the hybrid algorithm was approximately 3.4 CPU minutes on the Cray YMP for 1000 iterations. It can be seen that the two results are in good agreement. It is believed that discrepancies are due to ABC wall placement and the fact that a rectangular-waveguide feed rather than a dipole feed was used in the measurement.

V. COMMENTS ON THE EFFICIENCY OF THE YEE/SCALAR-WAVE ALGORITHM

Although partitioning the problem into planar regions is always possible, an important factor in determining how much computation time and memory will be saved by a hybrid-FDTD formulation is the relative volume of the divergence-free region compared to that of the nondivergence-free region. To minimize the computation and memory, the volume of the nondivergence-free regions should be made as small as possible to maximize the use of the more economical scalar-wave formulation. Suitable geometries include planar geometries in which nondivergence-free regions are often localized to thin planar volumes. In such cases, a hybrid Yee algorithm/scalar-wave algorithm can be expected to result in a memory and computational savings approximately equivalent to that of a FDTD scalar-wave algorithm (based on two components) over the entire domain.

Because a significant reduction in computation and memory is created by omitting the computation of fields, a potential drawback of using the proposed hybrid scheme is the inevitable loss of electric- or magnetic-field information. This drawback is somewhat offset by the fact that total field information, e.g., to compute the electric or magnetic current, is often required at or around the nondivergence-free regions such as pec, pmc, or dielectric surfaces where the Yee algorithm must be applied anyway as part of the hybrid scheme. However, if information about all six field components is needed in the divergence-free regions, e.g., to compute the power crossing a plane, the Yee algorithm can be applied locally where needed with minimal computational overhead.

VI. CONCLUSIONS

A hybrid Yee algorithm/scalar-wave equation formulation that is mathematically and numerically equivalent to the conventional Yee algorithm has been proposed. Limitations on its application as well as the applicability of the absorbing boundary conditions were also discussed. It is found that this hybrid approach is a viable way to reduce the memory by 33% and the computation by approximately 50% over the standard Yee algorithm. In addition, a method of applying the absorbing boundary condition on the normal fields (relative to an ABC wall) using the divergence relation in conjunction with the vector-wave equation is investigated and found to yield results numerically identical with the conventional application of the ABC based on the tangential fields. Numerical results generated by the hybrid Yee/scalar-wave algorithm for the Vivaldi antenna are given and compared with published experimental work.

APPENDIX A

We wish to show that a solution to the initial value problem of the wave equation, i.e.

$$\nabla^2 \bar{E}(\bar{r},t) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\bar{E}(\bar{r},t) = 0$$

where c = constant, will preserve the divergence relation, $\nabla \cdot \bar{E}(\bar{r},t) = 0$ for $t \geq t_0$ if $\nabla \cdot \bar{E}(\bar{r}, t = t_0) = 0$.

Proof: Taking the divergence of the wave equation, we obtain

$$\nabla^2 \phi(\bar{r},t) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\phi(\bar{r},t) = 0 \tag{A.1}$$

where $\phi(\bar{r},t) = \nabla \cdot \bar{E}(\bar{r},t)$.

If the fields are divergence-free at some arbitrary point in time, t_0, i.e.,

$$\nabla \cdot \bar{E}(\bar{r}, t = t_0) = 0 \tag{A.2}$$

it can be concluded by inspection that the solution to (A.1) is given by

$$\phi(\bar{r},t) = \nabla \cdot \bar{E}(\bar{r},t) = 0 \quad \text{for } t \geq t_0 .$$

Note: A similar proof involving the finite-difference approximation of the wave equation can be done using the finite-difference analog of the continuous-vector operations.

APPENDIX B

We want to prove that the Maxwell's curl equations (source-free) given by

$$\nabla \times \bar{E}(\bar{r},t) = -\mu \frac{\partial \bar{H}(\bar{r},t)}{\partial t}$$
$$\nabla \times \bar{H}(\bar{r},t) = \varepsilon \frac{\partial \bar{E}(\bar{r},t)}{\partial t}$$

will preserve the divergence-free nature of the electric and magnetic field for $t \geq t_0$.

Proof: Taking the divergence of both equations, one obtains the following equations

$$\frac{\partial \phi_h(\bar{r},t)}{\partial t} = 0 \tag{B.1}$$
$$\frac{\partial \phi_e(\bar{r},t)}{\partial t} = 0 \tag{B.2}$$

where $\phi_h(\bar{r},t) = \nabla \cdot \bar{H}(\bar{r},t)$ and $\phi_e(\bar{r},t) = \nabla \cdot \bar{E}(\bar{r},t)$.

If the fields are divergence-free at an arbitrary point in time, t_0, i.e.

$$\phi_h(\bar{r},t=t_0) = \nabla \cdot \bar{H}(\bar{r},t=t_0) = 0 \tag{B.3}$$
$$\phi_e(\bar{r},t=t_0) = \nabla \cdot \bar{E}(\bar{r},t=t_0) = 0 \tag{B.4}$$

it can be concluded by inspection that the solution to (B.2) and (B.4) implies that

$$\nabla \cdot \bar{H}(\bar{r},t) = \nabla \cdot \bar{E}(\bar{r},t) = 0 \quad \text{for } t \geq t_0 .$$

Note: A similar proof for the Yee algorithm can be done using the finite-difference analog of the vector operations.

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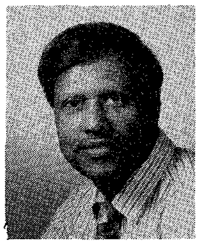
Jin-Fa Lee (M'88) was born in Taipei, Taiwan, in 1960. He received the B.S. degree from National Taiwan University, in 1982, and the M.S. and Ph.D. degrees from Carnegie-Mellon University, Pittsburgh, PA, in 1986 and 1989, respectively, all in electrical engineering.

From 1988 to 1990, he was with ANSOFT Corp., where he developed several CAD/CAE finite element programs for modeling three-dimensional microwave and millimeter-wave circuits. From 1990 to 1991, he was a post-doctoral fellow at the University of Illinois at Urbana-Champaign. Currently, he is an Assistant Professor at the Department of Electrical and Computer Engineering, Worcester Polytechnic Institute, Worcester, MA. His current research interests are: analyses of numerical methods, coupling of active and passive components in high-speed electronic circuits, three-dimensional mesh generation, and nonlinear optic fiber modelings.



Raj Mittra is the Director of the Electromagnetic Communication Laboratory of the Electrical and Computer Engineering Department and Research Professor of the Coordinated Science Laboratory at the University of IL, Urbana-Champaign. He has been a Visiting Professor at Oxford University, Oxford, England, and at the Technical University of Denmark, Lyngby. His professional interests include the areas of computational electromagnetics, electromagnetic modeling of electronic packaging, radar scattering, satellite antennas, microwave and millimeter wave integrated circuits, frequency selective surfaces, EMP and EMC analysis, and remote sensing. He has published over 350 journal papers and 20 book or book chapters on various topics related to electromagnetics. Mr. Mittra is a Past-President of AP-S and he has served as the Editor of the TRANSACTIONS of the Antennas and Propagation Society. He won the Guggenheim Fellowship Award in 1965 and the IEEE Centennial Medal in 1984. Currently, he serves as the North American Editor of the journal AEÜ. He is President of RM Associates, which is a consulting organization providing services to several industrial and governmental organizations.